Average BER Analysis for Binary Signallings in
Decode-and-Forward Dissimilar Cooperative Diversity Networks

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Abstract—In this letter, the average bit-error rate (BER) performance is analyzed for uncoded decode-and-forward (DF) cooperative diversity networks. We consider two typical networks: a single-relay cooperative network with the direct source-destination link and a two-relay cooperative network with the direct source-destination link, under dissimilar network settings, i.e., the fading channels of different relay branches may have different variances. We first derive a closed-form approximate average BER expression of binary signallings including noncoherent binary frequency shift keying (BFSK), coherent BFSK, and coherent binary phase shift keying (BPSK) for the single-relay network. We then generalize our analysis to the two-relay network, and a closed-form approximate average BER expression for binary signallings is derived. We also show that our BER expressions can be considered as generalizations of previously reported results in the literature. Throughout our analysis, only one approximation, so-called the piecewise-linear approximation, is made. Simulation results are in excellent agreement with the theoretical analysis, which validates our proposed BER expressions.

Index Terms—Binary signallings, bit-error rate (BER), decode-and-forward (DF), dissimilar networks, piecewise-linear (PL) approximation.

I. INTRODUCTION

DIVERSITY has been acknowledged as one of the most effective techniques to combat fading effects in wireless communications [1]. Recently, a new kind of diversity, cooperative diversity [2]–[5], has been proposed as a means of attaining broader cell coverage, mitigating channel impairments, and increasing the channel capacity without using multiple antennas at each terminal. One of the most well-known cooperative protocols is the decode-and-forward (DF) protocol, in which each relay receives and decodes the signal transmitted by the source and then forwards the decoded signal to the destination. There has been a lot of works carried out on the DF protocol incorporated with channel codes, for which the name coded cooperation is given [6]–[8]. Some bit-error rate (BER) and frame-error rate bounds have been derived in [6]–[8]. On the other hand, many works have focused on uncoded DF cooperation where no channel codes are used [9]–[13].

Recently, many researchers have started to analyze the performance of ML detection for uncoded DF cooperation [10]–[13]. However, the exact closed-form average BER analysis is extremely difficult even for single-relay cooperative networks with binary modulations [9], [10]. Therefore, most of the works have been devoted to the approximate average BER analysis. Nonetheless, still very limited results have been reported in the literature. In [10], Chen and Laneman have proposed an accurate approximation, which was referred to as the piecewise-linear (PL) approximation, and they obtained an accurate approximate closed-form average BER expression for single-relay systems with noncoherent binary frequency shift keying (BFSK). For single-relay systems with coherent BFSK, two approximate average BER expressions have been derived in [12] and [13] based on the PL approximation; however, the BER expressions are expressed in double-integral forms, which are not truly closed-form in a strict sense and are quite difficult to use in practice. For single-relay systems with coherent BFSK, no average BER expression has been reported, and it has been acknowledged that it is hard to analyze the average BER performance of the ML detection in closed-form for coherent BFSK even for single-relay cooperative networks [10]. For two-relay cooperative networks, only one BER expression has been reported in the literature [11], eq. (4.14)]. However, this expression is only applicable for noncoherent BFSK and for symmetric networks where the channel variances of the first hop for different branches are equal. However, the symmetric settings is not practical as the author acknowledged [11].

To the best of our knowledge, no closed-form average BER expressions of coherent binary signallings for single-relay uncoded DF cooperative networks have been found, and no closed-form BER expressions of binary signallings for two-relay uncoded DF dissimilar cooperative networks have been found either. This motivated our work.

In this letter, we analyze the average BER performance of the ML detection for uncoded DF cooperative networks with dissimilar settings. Specifically, we consider two typical cooperative diversity networks: the single-relay cooperative diversity network [10]–[13] and the two-relay cooperative diversity network [11], [14], [15]. First, we derive the probability density functions (PDFs) and cumulative distribution functions
(CDFS) of the sufficient statistics for the ML decision-making at the destination. Then we apply the accurate PL approximation and derive closed-form approximate average BER expressions with the help of the obtained PDFs and CDFS. Our BER expressions are shown to be valid for the general dissimilar uncoded DF networks adopting both coherent and noncoherent binary signalings. We also show that our BER expressions can be considered as generalizations of the previously reported results in the literature. The numerical results demonstrate that our BER expressions are extremely accurate.

The remainder of this letter is organized as follows. In Section II, we describe the system model and review the ML detection for uncoded DF networks. In Section III, we derive a closed-form approximate average BER expression for single-relay uncoded DF networks. In Section IV, a closed-form approximate average BER expression for two-relay uncoded DF networks is proposed. Section V presents some numerical results and Section VI concludes this letter.

Notation: For a real-valued random variable \( X \), \( X \sim \mathcal{N}(\mu_x, \Omega_x) \) indicates that \( X \) is a real-valued Gaussian random variable with mean \( \mu_x \) and variance \( \Omega_x \). For a complex-valued random variable \( Z \), \( Z \sim \mathcal{CN}(\mu_z, \Omega_z) \) indicates that \( Z \) is a circularly symmetric complex-valued Gaussian random variable with mean \( \mu_z \) and variance \( \Omega_z \).

II. SYSTEM MODEL

Consider a cooperative diversity network consisting of a source, \( K \) relays, and a destination as shown in Fig. 1. The source and the destination are denoted by terminal 0 and terminal \( K + 1 \), respectively, and relays are denoted by terminal \( i \), \( i = 1, 2, \cdots, K \). In this letter, we consider the uncoded cooperation with DF protocol, i.e., no channel codes are used. Each terminal in the network is equipped with a single antenna working in the half-duplex mode. We consider an orthogonal transmission scheme in which only one terminal is allowed to transmit at each time slot [4], [9], [10]. Therefore, the data transmission consists of two phases. In the first phase, the source broadcasts the signal, and each relay attempts to decode the signal. In the second phase, different relay terminals transmit their own remodulated signals in different time slots.

The channel coefficients \( h_{i,j} \) for the link between terminal \( i \) and terminal \( j \) are modeled as mutually independent complex Gaussian random variables with \( h_{i,j} \sim \mathcal{CN}(0, \sigma_{ij}^2) \), \( i, j \in \{0, 1, 2, \cdots, K + 1\} \). The additive noise associated with \( h_{i,j} \) is denoted by \( n_{i,j} \) for BPSK and denoted by \( n_{i,j,k} \), \( k \in \{0, 1\} \), for BFSK, where the third subscript \( k \) is the index of the two frequency subbands for BFSK signalling. We model the noise terms as mutually independent additive white Gaussian noise (AWGN) with zero mean and variance \( N_0 \). Consequently, the instantaneous signal-to-noise ratio (SNR) \( \gamma_{i,j} \) of the channel from terminal \( i \) to terminal \( j \) is given by \( \gamma_{i,j} = E_i|h_{i,j}|^2/N_0 \) and the average SNR is given by \( \bar{\gamma}_{i,j} = E_i\sigma_{ij}^2/N_0 \), where \( E_i \) is the average transmission power at terminal \( i \).

For BPSK signalling, the signals received by terminal \( i \), \( i = 1, 2, \cdots, K + 1 \), which were transmitted from the source, through the first frequency subband \( y_{0,i,0} \) and through the second frequency subband \( y_{0,i,1} \) are given by

\[
\begin{align*}
y_{0,i,0} &= (1 - x_0)\sqrt{E_0 h_{0,i}} + n_{0,i,0}, \\
y_{0,i,1} &= x_0 \sqrt{E_0 h_{0,i}} + n_{0,i,1},
\end{align*}
\]

where \( x_0 = 0 \) if the first frequency subband is used and \( x_0 = 1 \) if the second frequency subband is used.

For BPSK signalling, the signal \( y_{0,i} \) received by terminal \( i \), \( i = 1, 2, \cdots, K + 1 \), which was transmitted from the source, is given by

\[
\begin{align*}
y_{0,i} &= (1 - 2x_0)\sqrt{E_0 h_{0,i}} + n_{0,i},
\end{align*}
\]

where \( x_0 \in \{0, 1\} \).

In this letter, we will refer to \( x_0 \) as the transmitted signal at the source for BPSK and BPSK modulations, because \( x_0 \) can represent the two possibilities of the binary transmission. For DF, we suppose that, at relay \( i \), the signal \( x_0 \) is decoded into \( x_i \), \( x_i \in \{0, 1\} \). Similarly, we will refer to \( x_i \) as the transmitted signal at relay \( i \). Then for BFSK signalling, the signals received by the destination, which were transmitted from relay \( i \), through the two frequency subbands, are given by

\[
\begin{align*}
y_{i,K+1,0} &= (1 - x_i)\sqrt{E_i h_{i,K+1}} + n_{i,K+1,0}, \\
y_{i,K+1,1} &= x_i \sqrt{E_i h_{i,K+1}} + n_{i,K+1,1}.
\end{align*}
\]

For BPSK signalling, the signal \( y_{i,K+1} \) received by the destination, which was transmitted from relay \( i \), is given by

\[
\begin{align*}
y_{i,K+1} &= (1 - 2x_i)\sqrt{E_i h_{i,K+1}} + n_{i,K+1},
\end{align*}
\]

The log-likelihood ratio (LLR) of the ML detection for the uncoded DF cooperation with binary modulations has been shown to be given by [9], [10]:

\[
\text{LLR} = t_0 + \sum_{i=1}^{K} \psi(t_i),
\]

where

\[
\psi(t_i) = \ln \frac{(1 - \epsilon_i)e^{t_i} + \epsilon_i}{\epsilon_i e^{t_i} + (1 - \epsilon_i)}.
\]

In this equation, \( \epsilon_i \) represents the average BER\(^4\) at relay \( i \).

\(^4\)In this letter, we apply the same channel state information (CSI) assumption as in [10], [11] and [13] for coherent detection. Specifically, we assume that the destination only has the statistical CSI rather than the instantaneous CSI of the source-relay channels. Therefore, the destination only knows the average BER at the relays.
and it is given by
\[
\epsilon_i = \begin{cases} 
\frac{1}{2} \left( 1 - \sqrt{\frac{\lambda_i}{1 + \lambda_i}} \right), & \text{for coherent BFSK}, \\
\frac{1}{2} \left( 1 - \sqrt{\frac{\lambda_i}{1 + \lambda_i}} \right), & \text{for coherent BPSK}, \\
\frac{2 + \lambda_i}{2 + \lambda_i}, & \text{for noncoherent BFSK},
\end{cases}
\]
for \( i = 1, 2, \cdots, K \). The sufficient statistics \( t_i, i = 0, 1, \cdots, K + 1 \), are given by
\[
t_i = \begin{cases} 
\frac{2 \sqrt{T_i} \Re\{h_i^{*}K_{i+1}(y_i, K_{i+1}) - y_{i, K_{i+1}}(y_i, K_{i+1})\}}{4 \sqrt{T_i} \Re\{h_i^{*}K_{i+1}(y_i, K_{i+1})\}} - \frac{1}{y_i + 1}, & \text{for coherent BFSK}, \\
\frac{2 \sqrt{T_i} \Re\{h_i^{*}K_{i+1}(y_i, K_{i+1})\}}{4 \sqrt{T_i} \Re\{h_i^{*}K_{i+1}(y_i, K_{i+1})\}} - \frac{1}{y_i + 1}, & \text{for coherent BPSK}, \\
\frac{1}{1 + T_i}, & \text{for noncoherent BFSK}.
\end{cases}
\]

Proof: From the standard probabilistic analysis, PDF of \( Y \) can be derived by taking expectation over the conditional PDF of \( Y \) given \( X \), and CDF of \( Y \) can be derived by integrating the PDF of \( Y \).

Corollary 1: Suppose \( Y = |Z|X - |Z|^2 \), where \( X \sim \mathcal{N}(0, \sigma_x^2) \) and \( Z \sim \mathcal{CN}(0, \sigma_z^2) \) are independent random variables. Then the PDF and CDF of random variable \( Y \) are given by 
\[
f(a, b, c, y) = \begin{cases} 
a \exp \left( cy \right), & y \leq 0, \\
b \exp \left( by \right), & y > 0,
\end{cases}
\]
and the CDF \( F(a, b, c, y) \) of random variable \( Y \) is given by
\[
F(a, b, c, y) = \begin{cases} 
\frac{a}{a+b} \exp \left( cy \right), & y \leq 0, \\
\frac{a}{a+b} \exp \left( by \right), & y > 0,
\end{cases}
\]
where the parameters \( a, b, \) and \( c \) are given by
\[
a = \frac{1}{\sigma_x^2 + 2\sigma_z^2}, \\
b = \frac{1}{\sigma_z^2} \left( 1 - \frac{\sigma_x^2 + 2\sigma_z^2}{\sigma_z^2} \right), \\
c = \frac{1}{\sigma_z^2} \left( 1 + \frac{\sigma_x^2 + 2\sigma_z^2}{\sigma_z^2} \right).
\]
$t_j$ are two different statistics taken from (8) for a common modulation scheme with $i,j \in \{0,1,\ldots,K\}$ and $i \neq j$. Define $b_j = (1-x_j)b_j - x_jc_j$ and $c_j = (1-x_j)c_j - x_jb_j$.

Then $g_{x_j}(\cdot)$ is given as follows: If $b_i \neq b_j$ and $c_i \neq c_j$,

$$g_{x_j}(a_i,b_i,c_i,a_j,b_j,c_j,S_1,S_2)$$

$$= \left\{ \begin{array}{l}
\frac{a_i(b_i-c_i)}{b_i} \left[ F(a_j,\hat{b}_j,\hat{c}_j,2) - F(a_j,\hat{b}_j,\hat{c}_j,-S_2) \right] \\
+ \frac{a_i(a_j,exp(b_i,S_1))}{b_i(b_i-c_i)} \left[ \exp\left[\left(S_1 - S_2 - 1\right)\left(b_i - c_i\right)\right]\right] \\
+ \frac{a_i(a_j,exp(b_i,S_1))}{b_i(b_i-c_i)} \left[ 1 - \exp\left[\left(b_i - c_i\right)S_2\right]\right], \\
 \text{for } S_1 > S_2 > 0,
\end{array} \right.$$ (15)

If $b_i = \hat{b}_i$, the last terms in the first and second cases of (15) are replaced by $\frac{a_i(a_j,exp(b_i,S_1))}{b_i}$ and $\frac{a_i(a_j,exp(b_i,S_1))}{b_i}$, respectively. If $c_i = \hat{c}_j$, the last terms in the third and fourth cases of (15) are replaced by $\frac{a_i(a_j,exp(b_i,S_1))}{b_i}$ and $\frac{a_i(a_j,exp(b_i,S_1))}{b_i}$, respectively.

**Proof:** By Theorem 1 and performing some mathematical manipulations, it is not hard to prove this lemma.

From the expressions of $a_i$, $b_i$, and $c_i$ given in Theorem 1, it can be easily verified that $a_i > 0$, $b_i < 0$, and $c_i > 0$. By some simple manipulations, it can be easily verified that $a_i \neq c_i$, $b_i + c_i \neq 0$, $b_i - c_i \neq 0$, and $c_i - b_i \neq 0$ for any $i,j \in \{0,1,\ldots,K\}$; but it is possible that $b_i = \hat{b}_j$, $c_i = \hat{c}_j$, or $b_i + c_i = 0$ for $i,j \in \{0,1,\ldots,K\}$ and $i \neq j$. Therefore, the possibilities $b_i = \hat{b}_i$ and $c_i = \hat{c}_j$ need to be considered, and the possibilities $b_i - c_i = 0$ and $c_i - b_i = 0$ need not be considered in Lemma 2.

With Theorem 1 and Lemma 2, in the following theorem, we now derive an average BER expression of single-relay systems.

**Theorem 2:** For binary modulations, a closed-form approximate average BER $P_{B,1}$ of the single-relay ($K = 1$) cooperative diversity network is

$$P_{B,1} = \left\{ \begin{array}{l}
\left(1 - \epsilon_1\right) \left\{ g_0(a_0,b_0,c_1,0,T_1) \\
+ F(a_0,b_0,c_1,T_1) \left[ 1 - F(a_1,b_1,c_1,T_1) \right] \\
+ F(a_0,b_0,c_1,T_1)F(a_1,b_1,c_1,-T_1) \right\} \\
+ \epsilon_1 \left\{ g_1(a_0,b_0,c_1,0,T_1) \\
+ F(a_0,b_0,c_1,-T_1)F(a_1,b_1,c_1,-T_1) \right\}, \\
\end{array} \right.$$ (16)

where $\epsilon_1$, $T_1$, $a_i$, $b_i$, and $c_i$, $i = 0, 1$, are modulation-dependent parameters as described before.

**Proof:** Applying the total probability theorem, Theorem 1, and Lemma 2 yields this theorem.

Note that (16) is truly a closed-form approximate average BER expression, which does not involve any numerical computations. In the expression of (16), by choosing $a_i$, $b_i$, and $c_i$ as described in Theorem 1, $\epsilon_1$ as given by (7), and $T_1 = \ln\epsilon_1^{-1}$, we obtain the closed-form approximate average BERs of single-relay cooperative diversity networks for different binary modulation schemes including noncoherent BFSK, coherent BFSK, and coherent BPSK. In [10, eq. (14)], a closed-form approximate average BER expression was reported for the single-relay network with noncoherent BFSK. By simple manipulations, it can be shown that our BER expression of (16) reduces to eq. (14) of [10]. In this sense, therefore, our BER expression of (16) can be considered as a generalization of the expression of [10]. For coherent BPSK with a single relay, two approximate average BER expressions have been derived based on the PL approximation [12], [13]. However, these expressions involve numerical integrations, and thus, they are not truly closed-form. On the other hand, with the same PL approximation, our BER expression provides a truly closed-form approximate average BER expression for coherent BPSK. For coherent BFSK with a single relay, no average BER expression has been reported in the literature (see footnote 3). Therefore, our expression of (16) is the closed-form approximate average BER expression reported in the literature for the first time for coherent BFSK. Numerical results in Section V will demonstrate that (16) provides extremely accurate error probabilities for the ML detection in single-relay cooperative diversity networks adopting both coherent and noncoherent binary signalings.

In the next section, we will extend our analysis to the two-relay cooperative diversity networks.

**IV. CLOSED-FORM APPROXIMATE AVERAGE BER FOR TWO-RELAY NETWORKS**

In this section, we consider the two-relay dissimilar cooperative diversity network in which the channel variances of different relay branches are different in general. We derive a closed-form approximate average BER expression for the two-relay DF cooperative network again based on the PL approximation. Recall that the BER performance analysis of the single-relay case involved the computation of the probability for two statistics $t_0$ and $t_1$. For the two-relay case, one more statistic $t_2$ is involved, and therefore, the BER performance
TABLE I

<table>
<thead>
<tr>
<th>$x_j$</th>
<th>$x_k$</th>
<th>$A_1(x_j,x_k)$</th>
<th>$B_1(x_j,x_k)$</th>
<th>$A_2(x_j,x_k)$</th>
<th>$B_2(x_j,x_k)$</th>
<th>$A_3(x_j,x_k)$</th>
<th>$B_3(x_j,x_k)$</th>
<th>$A_4(x_j,x_k)$</th>
<th>$B_4(x_j,x_k)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>$[1 - F(a_k, b_j, c_j, T_j) - F(a_k, b_k, c_j, T_k)]$</td>
<td>$[1 - F(a_k, b_k, c_j, T_k)]$</td>
<td>$[1 - F(a_k, b_j, c_j, T_j) - F(a_k, b_k, c_j, T_k)]$</td>
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<td>1</td>
<td>$[1 - F(a_k, b_j, c_j, T_j)]$</td>
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<td>$F(a_k, b_k, c_j, T_k)$</td>
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</table>

Proof: We can prove this lemma with the help of Theorem 1 and Lemma 2; but the proof is very long. Thus, we will just present the main ideas for the proof. Since three random variables $t_i, t_j,$ and $t_k$ are involved in the definition of $W_{x_j, x_k}(\cdot),$ a triple-integral needs to be solved. Because of the asymmetry of the PDF of $t_i$ given $x_i,$ it is needed to divide the integral region for each integral such that a fixed PDF function is determined in each integrand. After dividing the integral regions according to this requirement, the expression of $W_{x_j, x_k}(\cdot)$ can be derived through some simple manipulations.

For the same reason as in Lemma 2, only the possibilities $b_i = b_j$ and $b_i = b_k$ are considered for (18), and $c_i = c_j$ and $c_i = c_k$ are considered for (19) in Lemma 3. Using Lemma 3, we can prove the following lemma.

**Lemma 4:** Let $t_i, t_j,$ and $t_k$ be three different statistics taken from (8) for a common modulation scheme with $i, j, k \in \{0, 1, \cdots, N\}$ and $i \neq j \neq k,$ $x_j, x_k \in \{0, 1\}.$ Define $\Pr(t_i + \psi_{PL}(t_j) + \psi_{PL}(t_k) < 0 | x_i, x_j, x_k) = U_{x_j, x_k}(a_i, b_i, c_i, a_j, b_j, c_j, a_k, b_k, c_k, T_j, T_k)$ assuming $0 \leq T_j \leq T_k.$ Then

\[
U_{x_j, x_k}(a_i, b_i, c_i, a_j, b_j, c_j, a_k, b_k, c_k, T_j, T_k) = A_{1,1}(x_j, x_k)F[a_i, b_i, c_i, -T_j + T_k] + A_{2,1}(x_j, x_k)F[a_i, b_i, c_i, -T_j + T_k] + A_{3,1}(x_j, x_k)F[a_i, b_i, c_i, -T_j + T_k]
\]

and the coefficients $A_{i,1}(x_j, x_k)$ and $B_{i,1}(x_j, x_k), i = 1, 2, 3, 4,$ are presented in Table I.

Proof: Applying the total probability theorem according to the possibilities of $t_j$ and $t_k,$ $U_{x_j, x_k}(\cdot)$ can be written as a combination of nine terms. We observed that the nine terms can be classified as three types: the first four terms including the probability of a single random variable, the following four terms including the joint probability of two random variables, and the last including the joint probability of three random variables. Therefore, the first four terms can be determined by the individual CDFs (11) of $t_i, t_j,$ and $t_k,$ given $x_i, x_j,$ and $x_k,$ respectively. The following four terms can be determined through Lemma 2. The last term can be determined through Lemma 3. Finally, the expression of (20) can be derived.

Using Lemma 4, we now derive an average BER expression for the two-relay cooperative networks in the following theorem.

**Theorem 3:** For binary modulations, a closed-form approximate average BER $P_{B,2}$ of the two-relay $(K = 2)$ cooperative

In this equation, $m = x_j + x_k,$ where $\oplus$ represents the modulo 2 addition. Function $\eta_{x_j, x_k}(\cdot)$ is defined as follows:

If $b_i \neq b_j$ and $b_i \neq b_k,$ $\eta_{x_j, x_k}(\cdot)$ is given on top of the next page: If $b_i = b_j$, the second term of (18) is replaced by $\eta_{a_i, a_j} = \frac{a_i a_j}{b_i} \exp((b_i - c_i)T_j - 1) T_j \exp((b_i - c_i)T_j).$ If $b_i = b_k,$ the last term of (18) is replaced by $\eta_{a_i, a_k} = \frac{a_i a_k}{c_i} \exp((b_i - c_i)T_j - 1) \exp((b_i - c_i)T_j).$ Also, function $\varphi_{x_j, x_k}(\cdot)$ is defined as follows:

If $c_i \neq c_j,$ $c_i \neq c_k,$ $\varphi_{x_j, x_k}(\cdot)$ is given on top of the next page: If $c_i = c_j,$ the second term of (19) is replaced by $\frac{a_i a_j}{c_i (b_i - c_i)} T_j \exp((b_i - c_i)T_j + 1 \exp((b_i - c_i)T_j).$ If $c_i = c_k,$ the last term of (19) is replaced by $\frac{a_i a_k}{c_i (b_i - c_i)} T_j \exp((b_i - c_i)T_j + 1 \exp((b_i - c_i)T_j).$
the expression of (21).

\[ n_{x_j, x_k}(a_i, b_i, c_i, a_j, b_j, c_j, a_k, b_k, c_k, T_j, T_k) \]
\[ = \frac{a_i a_j a_k}{b_i (\hat{c}_j - b_i)(\hat{c}_k - b_i)} \left\{ 1 - \exp[(b_i - \hat{c}_j)T_j] \right\} \left\{ 1 - \exp[(b_i - \hat{c}_k)T_k] \right\} \]
\[ + \frac{a_i a_j a_k}{b_i (\hat{c}_j - b_i)} \left\{ \exp[(b_i - \hat{c}_j)T_j - 1] \right\} \frac{\exp[(b_i - \hat{c}_j)T_j]}{b_j - \hat{c}_j} - \frac{1 - \exp[(b_i - \hat{c}_j)T_j]}{b_j - \hat{c}_j} \]
\[ + \frac{a_i a_j a_k}{b_i (\hat{c}_j - b_i)} \left\{ 1 - \exp[(b_i - \hat{c}_j)T_j] \right\} \frac{\exp[(b_i - \hat{c}_j)T_j]}{b_j - \hat{c}_j} - \frac{1 - \exp[(b_i - \hat{c}_j)T_j]}{b_j - \hat{c}_j} \right\} \] (18)

\[ \varphi_{x_j, x_k}(a_i, b_i, c_i, a_j, b_j, c_j, a_k, b_k, c_k, T_j, T_k) \]
\[ = \frac{a_i a_j a_k}{c_i (b_j - c_i)(b_k - c_i)} \left\{ \exp[(b_i - c_i)T_k - 1] \right\} \left\{ \exp[(b_k - c_i)T_k - 1] \right\} \]
\[ + \frac{a_i a_j a_k}{c_i (b_k - c_i)} \left\{ \exp[(b_k - c_i)T_k] \left\{ 1 - \exp[(c_i - \hat{c}_j)T_j] \right\} \right\} \frac{1 - \exp[(b_k - \hat{c}_j)T_j]}{\hat{c}_j - b_k} - \frac{1 - \exp[(b_k - \hat{c}_j)T_j]}{\hat{c}_j - b_k} \]
\[ + \frac{a_i a_j a_k}{c_i (\hat{c}_k - c_i)} \left\{ \exp[(b_i - c_i)T_j - 1] \right\} \frac{\exp[(b_i - \hat{c}_k)T_j]}{b_j - \hat{c}_k} - \frac{1 - \exp[(b_i - \hat{c}_k)T_j]}{b_j - \hat{c}_k} \right\} \] (19)

The expression of (21) can be considered as a generalization of the BER expression for the general dissimilar two-relay networks with coherent BFSK. In this sense, therefore, our BER expression of (21) can be considered as a generalization of the BER expression of [11]. For coherent BFSK and coherent BPSK, no average BER expressions have been reported for two-relay systems. Therefore, our expression of (21) is the closed-form approximate average BER expression reported in the literature for the first time for coherent BFSK and coherent BPSK. Numerical results in Section V will demonstrate that (21) provides extremely accurate error probabilities for the ML detection in two-relay dissimilar cooperative diversity networks adopting both coherent and noncoherent binary modulations.

V. NUMERICAL RESULTS

In this section, we compare the proposed approximate BER expressions with the exact BER obtained by Monte-Carlo simulations of the exact ML detection of (5) along with (6). We use the alphabetic indices \( \{ s, r, d \} \) to characterize the single-relay network, and the numeric indices \( \{ 0, 1, 2, 3 \} \) as in Section II to characterize the two-relay network. We consider an equal power allocation with total transmission power of the whole network normalized to 1. Specifically, \( E_s = E_r = 1/2 \) for single-relay systems and \( E_0 = E_1 = E_2 = 1/3 \) for two-relay systems. For ease of exposition, we assume that the source, relays, and the destination are located in a straight line.
\( P_{B.2} = (1 - \epsilon_{n1})(1 - \epsilon_{n2})Pr(t_0 + \psi pL(t_{n1}) + \psi pL(t_{n2}) < 0|x_0 = 0, x_{n1} = 0, x_{n2} = 0) + (1 - \epsilon_{n1})\epsilon_{n2}Pr(t_0 + \psi pL(t_{n1}) + \psi pL(t_{n2}) < 0|x_0 = 0, x_{n1} = 0, x_{n2} = 1) + \epsilon_{n1}(1 - \epsilon_{n2})Pr(t_0 + \psi pL(t_{n1}) + \psi pL(t_{n2}) < 0|x_0 = 0, x_{n1} = 1, x_{n2} = 0) + \epsilon_{n1}\epsilon_{n2}Pr(t_0 + \psi pL(t_{n1}) + \psi pL(t_{n2}) < 0|x_0 = 0, x_{n1} = 1, x_{n2} = 1). \)

\( (23) \)

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![Fig. 2. Average BER of the single-relay cooperative network with \( d_{s,r} = 0.1 \).](image)

![Fig. 3. Average BER of the single-relay cooperative network with \( d_{s,r} = 0.5 \).](image)

![Fig. 4. Average BER of the two-relay symmetric network with \( d_{0,1} = 0.5 \) and \( d_{0,2} = 0.5 \).](image)

![Fig. 5. Average BER of the two-relay dissimilar network with \( d_{0,1} = 0.1 \) and \( d_{0,2} = 0.9 \).](image)

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The distance \( d_{i,j} \) from node \( i \) to node \( j \) is normalized by the distance between the source node and the destination node, and hence, \( d_{s,r} + d_{r,d} = 1 \) for \( K = 1 \), and \( d_{0,1} + d_{0,3} = 1 \), \( i = 1, 2 \), for \( K = 2 \). The channel variances are modeled as \( \sigma^2_{i,j} = d_{i,j}^{-4} \). We plot the average BER with respect to the ratio of total power over the noise variance, which is \( 1/N_0 \).

Firstly, cooperative diversity networks with a single relay are considered. Figs. 2 and 3 show the results for cooperative networks with \( d_{s,r} = 0.1 \) and \( d_{s,r} = 0.5 \), respectively. Secondly, we consider cooperative diversity networks with two relays. Figs. 4 and 5 show the simulated average BER for two cases: 1) the symmetric network with \( d_{0,1} = 0.5, d_{0,2} = 0.5 \); 2) the dissimilar network with \( d_{0,1} = 0.1, d_{0,2} = 0.9 \). It can be seen that the proposed BER expressions overlap the simulation results. Moreover, the BER curves for both the single-relay and two-relay networks demonstrate 3 dB shifts to the left, from noncoherent BFSK to coherent BFSK and from coherent BFSK to coherent BPSK, which can be expected from coherent and noncoherent communication theories.

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5This assumption has also been made in [9]–[11] because it is very convenient to model the fading parameters for both dissimilar and symmetric cooperative diversity networks by simply choosing the positions of the relays in a straight line.
VI. CONCLUSIONS

In this letter, we have analyzed the average BER of the ML detection for uncoded DF cooperative diversity networks. Specifically, two typical cooperative diversity networks were considered: the single-relay cooperative diversity network with the direct source-destination link and the two-relay cooperative diversity network with the direct source-destination link. First, we derived the PDFs and CDFs of the sufficient statistics for the ML detection at the destination. Then we applied the accurate PL approximation and derived closed-form approximate average BER expressions with the help of the obtained PDFs and CDFs. Our BER expressions were shown to be valid for the general dissimilar DF networks adopting both coherent and noncoherent binary signalings. We also showed that our BER expressions can be considered as generalizations of the previously reported results in the literature. Throughout our analysis, only one approximation, i.e., the accurate PL approximation was made. Simulation results match excellently with the theoretical analysis, which validates our proposed BER expressions.

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