Error Probabilities of Noncoherent and Coherent FSK in the Presence of Frequency and Phase Offsets for Two-Hop Relay Networks

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Abstract—We analyze the error performance of two-hop relay networks adopting frequency shift keying (FSK) over frequency-flat Rayleigh fading channels. It is assumed that relay networks consist of a source, a relay, and a destination without a direct path signal from the source to the destination and the relay adopts the amplify-and-forward protocol with a fixed gain. Firstly, considering imperfect frequency and phase synchronization, we obtain the exact error probability expressions for noncoherent and coherent binary FSK (BFSK). Secondly, assuming perfect frequency and phase synchronization, we derive a closed-form error probability approximation for coherent M-ary FSK (MFSK). The proposed methods can also be used for the error performance analysis of classical one-hop FSK systems with perfect/imperfect frequency and phase synchronization. The obtained error probability expressions will help the design of two-hop relay networks adopting FSK in determining the system parameters such as the transmission power at the source, the amplifying coefficient at the relay, and the maximum affordable frequency and phase offsets to satisfy the required error performance.

Index Terms—Amplify-and-forward protocol, error probability, frequency and phase synchronization, frequency shift keying (FSK), two-hop relay networks.

I. INTRODUCTION

RELAY networks have received much attention in the literature because they can extend the system service coverage and increase the capacity [1]. This technique enables us to replace a single long-range direct transmission with multiple short-range low-power relay communications. The primary issues regarding relay networks for physical layer designers are power and cost. To accommodate low-power and low-cost constraints, it is essential to select a proper modulation scheme. Most of the previous works have dealt with full channel state information (CSI)-based modulations such as phase shift keying (PSK) and quadrature amplitude modulation (QAM) [2]–[5]. However, the full CSI-based modulations might not be a good choice for sensor networks, which require a small and simple hardware structure [6]. For such case, many researchers have considered orthogonal modulation schemes, especially frequency shift keying (FSK) [7]–[10], which requires no CSI for noncoherent detection and partial CSI, i.e. phase information, for coherent detection. This greatly reduces the burden on hardware. Furthermore, some works showed that binary FSK (BFSK) was more energy-efficient than M-ary FSK (MFSK) under start-up power dominant conditions [6], [7]. Thus, some publications on relay networks have mainly focused on BFSK modulation [8], [9].

Although FSK modulation has been regarded as a promising candidate for relay networks, the literature has paid very little attention to its error performance analysis. Hasna and Alouini derived the closed-form moment-generating function (MGF) for two-hop relay networks adopting the amplify-and-forward protocol with a fixed gain [3]. Having the MGF and using the MGF-based analysis in [11], it is possible to obtain the exact closed-form bit-error rate (BER) for noncoherent MFSK with perfect frequency synchronization and the exact BER in a one-integral form for coherent BFSK with perfect/imperfect frequency and phase synchronization. Chen and Laneman provided tight BER approximations for two-hop cooperative networks adopting BFSK and the decode-and-forward protocol [9]. Karagiannidis et al. presented a closed-form lower bound of BER for multi-hop relayed communications adopting coherent BFSK and the amplify-and-forward protocol [12]. To the best of our knowledge, however, there has been no BER expression in a simple form for relay networks adopting FSK with imperfect frequency and phase synchronization.

In this letter, we analyze the error performance of two-hop relay networks adopting noncoherent and coherent FSK over frequency-flat Rayleigh fading channels. It is assumed that relay networks consist of a source, a relay, and a destination without a direct path signal from the source to the destination and the relay adopts the amplify-and-forward protocol with a fixed gain. Firstly, a relay network adopting noncoherent FSK is considered. For BFSK with imperfect frequency synchronization, we obtain the exact error probability expression in a one-integral form. Secondly, a relay network adopting coherent FSK with partial CSI, i.e. phase information, is considered. We begin by deriving a closed-form error probability approximation for MFSK with perfect frequency and phase synchronization. Then we obtain the exact error probability expression in a simple form for BFSK with imperfect frequency and phase synchronization. Since classical one-hop FSK systems can be considered as a special case of two-hop relay networks adopting FSK, the proposed method can also be used for the performance analysis of classical one-hop FSK systems with perfect/imperfect frequency and phase synchronization.

Notation: We use $E_x[\cdot]$ to denote the expectation operation with respect to $x$. Also, $\Re\{x\}$ denotes the real part of a complex number $x$, and $x \sim \mathcal{CN}((\mu, \Omega))$ indicates that $x$ is a circularly symmetric complex Gaussian random variable with mean $\mu$. 

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and variance $\Omega$.

II. System Model

Consider a relay network consisting of a source, a relay, and a destination node without a direct path signal component where each node has a single antenna. There are $M$ orthogonal frequency subbands, and MFSK is used to transmit a block of $\log_2 M$ bits per symbol. In order to maintain orthogonality between adjacent frequencies, the minimum frequency separation is $\Delta f = 1/T$ for noncoherent FSK and $\Delta f = 1/(2T)$ for coherent FSK, where $T$ is the symbol duration. The waveform $s_k(t)$ transmitted by the source node at the first slot is given by

$$s_k(t) = \sqrt{\frac{2E_s}{T}} \cos(2\pi(f_c + (k - 1)\Delta f)t), \quad k = 1, \cdots, M,$$

where $E_s$ is the average energy per symbol and $f_c$ the carrier frequency. Let $h_1$ be the complex channel coefficient from the source to the relay node, which is fixed during a time slot with $h_1 \sim \mathcal{CN}(0, \Omega_1)$. When there exist frequency offset $f_{o,r}$ and phase offset $\theta_{o,r}$ at the relay, the received signal $r(t)$ at the relay node is given by

$$r(t) = \sqrt{\frac{2E_r}{T}} h_1 \cos(2\pi(f_c + (k - 1)\Delta f + f_{o,r} + \theta_{o,r})t + n(t),$$

where $n(t)$ is an additive white Gaussian noise (AWGN) process.

A relay node assists the communication from the source to the destination node using the amplify-and-forward protocol. A relay node multiplies the incoming signal with a fixed gain model (not a variable gain model) at the relay node. As in the previous publications [3], [13], if we select $\beta = \sqrt{E_r/(E_s + \sigma_n^2)}$ where $\sigma_n^2$ is the noise power at the relay, the average transmission power of the relay is $E_r$ in the long term irrespective of frequency offset and/or phase offset at the relay. Let $h_2$ denote the complex channel coefficient from the relay to the destination node, which is fixed during a time slot with $h_2 \sim \mathcal{CN}(0, \Omega_2)$. When there exist frequency offset $f_{o,d}$ and phase offset $\theta_{o,d}$ at the destination, the received signal $y(t)$ at the destination node is given by

$$y(t) = \sqrt{\frac{2E_s}{T}} \beta h_1 h_2 \cos(2\pi(f_c + (k - 1)\Delta f + f_{o,r} + \theta_{o,r})t + \theta_{o,d}) + \beta h_2 n(t - T) + v(t),$$

where $f_{o,r} = f_{o,d} + f_{o,dr}$, $\theta_{o} = \theta_{o,r} + \theta_{o,d}$, and $v(t)$ is an AWGN process at the destination. Let $n_m$ and $v_m$ denote the AWGNs of the $m$th subband correlator at the relay node and the destination node, respectively. They are given by $n_m = 1/\sqrt{E_s} \int_0^T n(t) s_m(t) dt$ and $v_m = 1/\sqrt{E_s} \int_0^T v(t) s_m(t - T) dt$, where $n_m \sim \mathcal{CN}(0, \sigma_n^2)$ and $v_m \sim \mathcal{CN}(0, \sigma_v^2)$. Then the output $y_{k,m} = 1/\sqrt{E_s} \int_T y(t) s_m(t - T) dt$ of the $m$th subband correlator at the destination node is given by

$$y_{k,m} = \begin{cases} \sqrt{E_s} \beta h_1 h_2 \delta(k - m) + \beta h_2 n_m + v_m, & \text{for noncoherent FSK with } f_o = 0 \text{ or } \theta_o = 0, \\ \sqrt{E_s} \beta h_1 h_2 c_{k,m} + \beta h_2 n_m + v_m, & \text{for noncoherent FSK with } f_o \neq 0, \\ \sqrt{E_s} \beta h_1 h_2 d_{k,m} + \beta h_2 n_m + v_m, & \text{for coherent FSK with } f_o \neq 0 \text{ or } \theta_o \neq 0, \end{cases}$$

where $c_{k,m} = \sin(2\pi f_o T)/(2\pi(k - m) + 2\pi f_o T)$, which is given in [14, eq. (3)], and $d_{k,m} = \sin(2\pi f_o T + \pi(k - m)/\cos(\theta_o - \sin(\theta_o)))/(\pi(k - m) + 2\pi f_o T)$. Also, $\delta(k - m) = 1$ for $k = m$, and $\delta(k - m) = 0$ for $k \neq m$. Note that for noncoherent and coherent FSK with perfect synchronization, $\{y_{k,m}\}_{m=1}^M$ are orthogonal to one another, whereas for noncoherent and coherent FSK with imperfect synchronization, $\{y_{k,m}\}_{m=1}^M$ may not be orthogonal.

III. Error Probability for Noncoherent FSK

When CSI is not available at the destination node, noncoherent detection schemes must be used. In this section, we consider the noncoherent FSK systems with imperfect frequency synchronization. The optimum receiver selects the signal with the maximum value from $M$ envelope detectors. In order to maintain orthogonality between adjacent frequencies, the minimum frequency separation is $\Delta f = 1/T$.

As acknowledged in the Introduction, using the closed-form MGF [3] and the MGF-based analysis in [11], it is possible to obtain the exact closed-form BER for MFSK with perfect frequency synchronization. Thus, we only focus on a realistic and practically important scenario, noncoherent BFSK with imperfect frequency synchronization.

A. Exact Error Probability Expression for BFSK with Imperfect Frequency Synchronization

In the case of imperfect frequency synchronization ($\Delta f \neq 0$), the symbols transmitted through different frequency subbands generally have different error probabilities. Thus, we first find the general error expression for the $k$th subband and then take the average over all possible subbands. The output $z_{k,m}$ of the $m$th envelope detector at the destination node is given by

$$z_{k,m} = |y_{k,m}| = \sqrt{E_s} \beta h_1 h_2 c_{k,m} + \beta h_2 n_m + v_m, \quad k, m = 1, 2.$$

Since $\{n_m\}_{m=1}^2$ and $\{v_m\}_{m=1}^2$ are independent complex Gaussian random variables, $\{z_{k,m}\}_{m=1}^2$ are independent random complex Gaussian random variables when $h_1$ and $h_2$ are fixed. The conditional symbol-error rate (SER) $P_{s|h_1,h_2}(k)$ is given by

$$P_{s|h_1,h_2}(k) = \Pr\{z_{k,k} < z_{k,3-k}|h_1, h_2\}.$$

Since $z_{k,k}$ and $z_{k,3-k}$ are independent random variables when $h_1$ and $h_2$ are fixed, using eq. (8.2.12) of [15],
\[ P_{s|h_1,h_2}(k) \text{ can be given by} \]
\[
P_{s|h_1,h_2}(k) = \frac{1}{2} \left[ 1 - Q_1(\sqrt{u_{k,m} t_1}, \sqrt{u_{k,3-k} t_1}) + Q_1(\sqrt{u_{k,3-k} t_1}, \sqrt{u_{k,m} t_1}) \right], \quad (6)
\]
where \( u_{k,m} = \sigma^2_{k,m} t_2 / (at_2 + b) \) with \( \sigma^2/E_s = b = \sigma^2/\beta^2 \) and \( t_1 = |h_1|^2 \) for \( i = 1, 2 \). Also, \( Q_1(\alpha_1, \alpha_2) \) is the first-order Marcum Q-function \([11, \text{eq. (4.34)}]\). Using eq. (5.52) of [11], \( P_{s|h_2}(k) = E_{h_2}[P_{s|h_1,h_2}(k)] \) can be given by
\[
P_{s|h_2}(k) = \frac{1}{2} - \frac{\nu_{k,k} - \nu_{k,3-k}}{4(1 + (\nu_{k,k} + \nu_{k,3-k})/2)^2 - \nu_{k,k}\nu_{k,3-k}}, \quad (7)
\]
where \( \nu_{k,m} = \Omega_1 u_{k,m} \). Then the SER \( P_s(k) = E_{h_2}[P_{s|h_2}(k)] \) can be given by
\[
P_s(k) = \int_0^\infty P_{s|h_2}(k) \frac{\exp(-t_2/\Omega_2)}{\Omega_2} dt_2. \quad (8)
\]
Finally, by substituting (8) into \([14, \text{eq. (18)}]\), we can obtain the exact BER expression in a one-integral form. Note that the obtained BER result can be used for the design of practical two-hop relay networks adopting noncoherent BFSK in determining the maximum affordable frequency offset to satisfy the required BER.

IV. ERROR PROBABILITY FOR COHERENT FSK

When partial CSI, i.e. phase information, is available at the destination node, coherent detection schemes can be used. In this section, we consider the coherent FSK systems with perfect/imperfect frequency and phase synchronization. The optimum receiver selects the signal with the maximum value from \( M \) phase coherent detectors. In this case, the minimum frequency separation is \( \Delta f = 1/(2T) \).

A. Closed-Form Error Probability Approximation for MFSK with Perfect Frequency and Phase Synchronization

When a symbol is transmitted through the first frequency subband at the source node, the real part output \( z_m \) of the \( n \)th phase coherent detector at the destination node is given by
\[
z_m = \Re[y_{1,m} e^{-j(\theta_1 + \theta_2)}] = \begin{cases} \sqrt{E_s} \beta_1 |h_1| |h_2| + \beta_1 |h_2| [\Re[r_{1,m} e^{-j(\theta_1 + \theta_2)}] + \Re[v_{1,m} e^{-j(\theta_1 + \theta_2)}]], & m = 1, \\ \beta_2 |h_2| [\Re[r_{m,m} e^{-j(\theta_1 + \theta_2)}] + \Re[v_{m,m} e^{-j(\theta_1 + \theta_2)}]], & m = 2, \cdots, M, \end{cases} \quad (9)
\]
where \( \theta_1 \) and \( \theta_2 \) are the phases of \( h_1 \) and \( h_2 \), respectively. Since \( \{r_{m,m}\}_{m=1}^M \) and \( \{v_{m,m}\}_{m=1}^M \) are independent complex Gaussian random variables, \( \{z_m\}_{m=1}^M \) are independent Gaussian random variables when \( h_1 \) and \( h_2 \) are fixed. Then the conditional probability density function \( p_{z_m|h_1,h_2}(x) \) can be given by
\[
p_{z_m|h_1,h_2}(x) = \begin{cases} \frac{\sqrt{2\pi}}{\sqrt{\lambda}} \exp(-x - \sqrt{E_s \beta_1^2 t_1^2/\lambda})/\lambda, & m = 1, \\ \frac{\sqrt{2\pi}}{\sqrt{\lambda}} \exp(-x^2/\lambda), & m = 2, \cdots, M, \end{cases} \quad (10)
\]
where \( \lambda = \beta^2 s_n^2 t_2 + \sigma^2 \).

From (10), the conditional SER \( P_{s|h_1,h_2} \) is given by
\[
P_{s|h_1,h_2} = 1 - \Pr\{z_1 > z_2, \cdots, z_M > z_M|h_1, h_2\} = 1 - \int_{-\infty}^\infty p_{z_1|h_1,h_2}(x) \left[ Q\left(-x \sqrt{2 \sigma^2 / \lambda}\right)\right]^{M-1} dx. \quad (11)
\]
Since we have no closed-form solution to the integration of (11), we use an approximation of the Q-function. Recently, a new and very accurate approximation of the Q-function was proposed [16]. However, it is very difficult to take expectation of the approximation over complex channel coefficients. Thus, we adopt an approximation of the Q-function given by a linear combination of two exponential functions [17]:
\[
Q(x) \approx \frac{1}{12} \exp(-x^2/2) + \frac{1}{4} \exp(-2x^2/3), \quad x \geq 0. \quad (12)
\]
Since (12) is only valid for the non-negative argument, we divide (11) into three parts:
\[
P_{s|h_1,h_2} = 1 - P_{s,1|h_1,h_2}(1, M - 1) - \sum_{l=0}^{M-1} (M - 1)^l P_{s,1|h_1,h_2}(2, l), \quad (13)
\]
where
\[
P_{s,1|h_1,h_2}(j, l) = \int_0^\infty p_{z_1|h_1,h_2}(l, j) \left[ Q\left(-\sqrt{\sigma^2 / \lambda}\right)\right]^{j} dx. \quad (14)
\]
By substituting (12) into (14), \( P_{s,1|h_1,h_2}(j, l) \) can be approximated into
\[
P_{s,1|h_1,h_2}(j, l) \approx \frac{1}{\sqrt{\lambda}} \int_0^\infty \exp\left(-((-1)^j x - \sqrt{E_s \beta_1^2 t_1^2/\lambda})^2/\lambda\right) \left[ \frac{1}{12} \exp\left(-x^2/\lambda\right) + \frac{1}{4} \exp\left(-4x^2/3\lambda\right)\right]^{j} dx. \quad (15)
\]
Taking a binomial expansion, (15) can be given by
\[
P_{s,1|h_1,h_2}(j, l) \approx \sum_{i=0}^{l} \binom{l}{i} \frac{3^j}{12^j} \left[ (-1)^{j-l} P_{s,1,1|h_1,h_2}(j, l) + (j-1) P_{s,1,2|h_1,h_2}(j, l) \right], \quad (16)
\]
where \( P_{s,1,1|h_1,h_2}(j, l) = Q\left(\sqrt{2E_s \beta_1^2 t_1^2/\lambda}\right) \exp(-g_1 E_s \beta_1^2 t_1^2 t_1^2/\lambda) \) and \( P_{s,1,2|h_1,h_2}(j, l) = \exp(-g_1 E_s \beta_1^2 t_1^2 t_1^2/\lambda) \) with
\[
g_1 = (l + j/3)/(l + 1 + j/3) \quad \text{and} \quad g_2 = 1/(l + 1 + j/3).
\]

For further derivation, we again use the Q-function approximation of (12). Actually, we can solve \( E_{h_1}[P_{s,1|h_1,h_2}(l, j)] \) in closed-form using MGF-based approach because all the integrands are exponential functions and Q-functions. However, it is extremely difficult to take an expectation of the closed-form expression over \( h_2 \). In this letter, therefore, we take another approach. Specifically, in order to obtain a closed-form SER expression, we again use the Q-function approximation of (12). Although we use the Q-function approximation twice, the final result is still very accurate, which will be numerically demonstrated in Section VI. Since \( E_{h_1}[\exp(-gt_1)] = (1 + \)
where $g_3 = (M + (i + 1)/3)/(M + i/3)$, $g_4 = (l + i/3)/(l + i/3)$, and $F_B(g) = (1 + gE_k^2\beta^2\Omega_1\tau_2)/\lambda$. Taking an expectation of $F_B(\gamma)$ over $h_2$ gives $F(\gamma) = \int_0^\infty F_B(\gamma) \exp(-t_2/\Omega_2)/\Omega_2 dt_2 = a_0/(a_0 + g + gb_0/(a_0 + g)) \exp(b_0/(a_0 + g)) E_1(b_0/(a_0 + g))$, where $a_0 = a/\Omega_1$, $b_0 = b/\Omega_1$, and $E_1(x) = \int_x^\infty \frac{1}{\tau} \exp(-\tau)d\tau$ is the exponential integral function. Then the final SER approximation $P_s = E_{h_2}[P_{s|h_2}]$, which is omitted here due to space limitation, can be easily obtained by replacing $F_B(\gamma)$ with $F(\gamma)$ in (17). Finally, by substituting the SER approximation $P_s$ into [11, eq. (8.68)], the closed-form SER approximation can be obtained.

Until now, we have assumed the perfect frequency and phase synchronization. In the next subsection, we consider a realistic and practically important scenario: coherent BFSK with imperfect frequency and phase synchronization.

**B. Exact Error Probability Expression for BFSK with Imperfect Frequency and Phase Synchronization**

When a symbol is transmitted through the $k$th frequency subband at the source node, the real part output $z_{k,m}$ of the $m$th frequency coherer detector at the destination node is given by

$$z_{k,m} = \Re[\beta_k e^{-j(\theta_1 + \theta_2)}]$$

$$= \sqrt{E_s} \beta_h [h_1 | h_2 | d_{k,m} + e^{-j\theta_2}] + \Re[v_{m} e^{-j(\theta_1 + \theta_2)}], \quad k, m = 1, 2.$$ (18)

Since $\{z_{k,m}\}_{m=1}^2$ are independent Gaussian random variables when $h_1$ and $h_2$ are fixed, the conditional SER $P_{s|h_1,h_2}(k)$ can be given by

$$P_{s|h_1,h_2}(k) = \Pr\left\{\sqrt{E_s} \beta [h_1 | h_2] < \beta [h_1 | (n_{k-3} - n_k) e^{-j\theta_2}] + \Re[(v_{k-3} - v_k) e^{-j(\theta_1 + \theta_2)}] | h_1, h_2\right\}$$

$$= Q(\beta \sqrt{T}),$$

where $\beta_k = d_{k,3-k} - d_{k,k-3}$ and $\gamma_T = t_1 t_2/(a_2 + b_2)$. Using the property $E_h[Q(\sqrt{g(h^2)})] = 0.5[1 - \sqrt{\gamma}]/(2 + \gamma)$ when $h \sim CN(0, \Gamma_1)$, $P_{s|h_1}(k) = E_{h_1}[P_{s|h_1,h_2}(k)]$ can be given by

$$P_{s|h_1}(k) = \frac{1}{2} \left(1 - (-1)^k \sqrt{e^{2z_{k,t_2}} / (2a_0 + \epsilon_k^2 z_{k,t_2} + 2b_1)}\right),$$ (19)

where $b_1 = b/\Omega_1$, $\zeta = 0$ for $\epsilon_k \geq 0$, and $\zeta = 1$ for $\epsilon_k < 0$. Using eq. (3.383.5) of [18], the SER $P_s(k)$ can be given by

$$P_s(k) = \frac{1}{2} - (-1)^k \sqrt{\frac{\sqrt{\pi e^2 b_0}}{2(2a_0 + \epsilon_k^2 z_{k,t_2})^{3/2}} \Psi \left(1.5, 2, \frac{2b_0}{2a_0 + \epsilon_k^2 z_{k,t_2}}\right),$$ (20)

where $\Psi(p, q, s) = \int_0^\infty \tau^{p-1}(1 + 1)^{q-1} \exp(-s\tau)d\tau$ is the confluent hypergeometric function of the second kind [18, eq. (9.211.4)]. As acknowledged in the Introduction, using the closed-form MGF [3] and the MGF-based analysis in [11], it is possible to obtain the exact SER in a one-integral form, which requires numerical computation due to the single integral. However, using (20), one can easily calculate SER without such numerical method. Finally, by substituting (20) into [14, eq. (18)], we can obtain the exact BER expression. As in the noncoherent BFSK with imperfect synchronization of Section III.A, the obtained SER result can be used to determine the maximum affordable frequency and phase offsets for the design of practical two-hop relay networks adopting coherent BFSK.

**Remark:** In Section IV.A, we obtained a very accurate closed-form error probability approximation for MFSK with perfect frequency and phase synchronization. With a simple modification of (20), we can derive the exact BER expression for BFSK with perfect frequency and phase synchronization. In this case, $\epsilon_k = 1$ and $\zeta = 0$. Thus, the final exact BER $P_b$ can be given by

$$P_b = \frac{1}{2} - \sqrt{\pi b_0} / (2a_0 + 1)^{3/2} \Psi \left(1.5, 2, \frac{2b_0}{2a_0 + 1}\right).$$ (21)

**V. SPECIAL CASES: CLASSICAL ONE-HOP FSK SYSTEMS**

Classical one-hop FSK systems can be considered as a special case of two-hop relay networks adopting FSK with a deterministic unit channel coefficient $h_2 = 1$ from the relay to the destination node and zero noise $\sigma^2 = 0$ at the destination node. In this case, $b_0 = 0$ and $a_0 = \sigma^2/(E_s^2\gamma_1) = 1/\gamma_0$ where $\gamma_0$ is the received SNR. In the following, we present some closed-form BER expressions for classical one-hop FSK systems which, to the best of our knowledge, have not been reported in the literature.

**A. Exact Closed-Form Error Probability Expression for Noncoherent BFSK with Imperfect Frequency Synchronization**

Since $v_{k,m} = d_{k,m}^2 \gamma_0$, (7) can be simplified as

$$P_s(k) = \frac{1}{2} - \frac{d_{k,k}^2 - d_{k,3-k}^2}{\sqrt{(2/\gamma_0 + (d_{k,k}^2 + d_{k,3-k}^2) - 4d_{k,k}^2d_{k,3-k})}}.$$ (22)

Then substituting (22) into [14, eq. (18)] gives the exact closed-form BER expression. Note that the obtained BER expression using (22) is a generalization of [19, eq. (10)] for a Rayleigh fading channel.
B. Closed-Form Error Probability Approximation for Coherent MFSK with Perfect Frequency and Phase Synchronization

Since $\mathcal{F}_B(g) = (1 + g\gamma_0)^{-1}$, (17) can be given by

$$P_s \approx 1 - \sum_{i=0}^{M-1} \left( \frac{M-1}{i} \right) \frac{3^i/12^{M-1}}{\sqrt{M + i/3}} \cdot \left( \frac{1}{12(1 + \gamma_0)} + \frac{1}{4(1 + g\gamma_0)} \right) - \sum_{l=0}^{M-1} \sum_{i=0}^{l} \left( \frac{M-1}{l} \right) \left( \frac{l}{i} \right) \frac{3^i/(-12)^l}{\sqrt{l + 1 + i/3}} \cdot \left( \frac{1}{1 - g\gamma_0} \right) \right).$$

Then substituting (23) into [11, eq. (8.68)] gives a closed-form BER approximation.

In the literature, the exact closed-form BER expressions have been obtained only for $M = 2, 3, 4$ [11, eqs. (8.132), (8.140), and (8.141)] and the exact BER expression in a one-integral form for $M > 4$ [20, eq. (8)], which requires numerical calculations. On the other hand, with (23), one can accurately calculate BER for $M > 4$ without any numerical method.

C. Exact Closed-Form Error Probability Expression for Coherent BFSK with Imperfect Frequency and Phase Synchronization

Since $a_0 = 1/\gamma_0$ and $b_1 = 0$, (19) can be simplified as

$$P_s(k) = \frac{1}{2} \left( 1 - (-1)^{\epsilon_0} \sqrt{\frac{\epsilon_2^2 \gamma_0}{2 + \epsilon_2^2 \gamma_0}} \right).$$

Then substituting (24) into [14, eq. (18)] gives the exact closed-form BER expression. Although it is not very difficult to derive (24), to the best of our knowledge, it has not been reported in the literature.

VI. SIMULATION RESULTS

This section gives some simulation results showing that the presented error probability expressions are very effective methods for BER evaluation of two-hop relay networks adopting noncoherent and coherent FSK modulations. We compare the BER values obtained by our analysis to those obtained by Monte Carlo simulations. Let $\gamma_1 = E_s/\Omega_1/\sigma_n^2$ and $\gamma_2 = \beta^2 E_s/\Omega_2/\sigma_c^2$. We let $\gamma_1$ and $\gamma_2$ change by varying $\sigma_n^2$ and $\sigma_c^2$, with fixed values $E_s = 1$, $\beta = 1$, and $\Omega_1 = \Omega_2 = 1$, and we assume that BFSK and 4FSK are used.

Firstly, we consider a two-hop relay network adopting noncoherent BFSK with imperfect frequency synchronization. Fig. 1 shows the average BER versus $f_o T$, normalized frequency offset, for a two-hop relay network adopting noncoherent BFSK with imperfect frequency synchronization when $\gamma_1 = \gamma_2 = 20$ dB and 30 dB. We can see that the BER performance deteriorates as $f_o T$ increases because the frequency of the received signal is shifted by the frequency offset $f_o$.

Secondly, we consider a two-hop relay network adopting coherent FSK with perfect/imperfect frequency and phase synchronization. Fig. 2 shows the average BER versus $\gamma_2$ with various $\gamma_1$ values for a two-hop relay network adopting coherent BFSK and 4FSK with perfect frequency and phase synchronization. For fixed values $\gamma_1$ = 15 dB and 25 dB, the performance does not linearly improve indefinitely even when $\gamma_2 \to \infty$. This is because the equivalent SNR, which is given by $\gamma_{eq} = \gamma_1 \gamma_2/(C + \gamma_2)$ with $C = E_s/\left(\beta^2 \sigma_n^2\right)$ [3, eq. (6)], remains finite and the performance will be saturated even if we increase $\gamma_2$ indefinitely. We can see the closed-form BER approximation using (17) is also very accurate. Fig. 3 shows the average BER versus $f_o T$, normalized frequency offset, for a two-hop relay network adopting coherent BFSK with imperfect frequency and phase synchronization when $\gamma_1 = \gamma_2 = 20$ dB with $\theta_o = -0.1$, $\gamma_1 = \gamma_2 = 30$ dB with $\theta_o = 0.1$, and $\gamma_1 = \gamma_2 = 40$ dB with $\theta_o = 0$. Again, we can see that the BER performance deteriorates as $f_o T$ increases for different values of the phase offset $\theta_o$. 
Finally, we investigate the effect of the relay location for a two-hop relay network adopting BFSK. Let $d_{S,R}$ denote the distance between the source and the relay and $d_{R,D}$ denote the distance between the relay and the destination, both of which are normalized by the distance between the source and the destination. Therefore, we have $d_{S,R} + d_{R,D} = 1$. We let the complex channel coefficient $h_1$ from the source to the relay be $h_1 d_{S,R}^\alpha$ and the complex channel coefficient $h_2$ from the relay to the destination be $h_2 d_{R,D}^{\alpha/2}$, where $h_1 \sim \mathcal{CN}(0,1)$, $h_2 \sim \mathcal{CN}(0,1)$, and $\alpha$ is the path loss exponent. Thus, we have $E[|\bar{h}_1|^2] = d_{S,R}^\alpha$ and $E[|\bar{h}_2|^2] = d_{R,D}^{\alpha}$. Fig. 4 shows the average BER versus $d_{S,R}$ for a two-hop relay network adopting noncoherent and coherent BFSK with perfect/imperfect frequency synchronization and perfect phase synchronization when $\gamma_1 = \gamma_2 = 30$ dB. We choose $\alpha = -4$. To ensure a fair comparison, we let the total transmission power of the source and the relay be constant. We can see that a network adopting coherent BFSK gives better BER performance than a network adopting noncoherent BFSK, both with perfect frequency synchronization. However, the latter is more robust to the frequency offset than the former because the minimum frequency separation of the latter is twice that of the former. Also, we have found that the relay location affected the error performance. With the obtained error probability expressions, it was also possible to analyze the error performance of classical one-hop FSK systems with perfect/imperfect frequency and phase synchronization.

VII. CONCLUSIONS

In this letter, we have analyzed the error performance of two-hop relay networks adopting noncoherent and coherent FSK over frequency-flat Rayleigh fading channels. It was assumed that relay networks consisted of a source, a relay, and a destination without a direct path signal from the source to the destination and the relay adopted the amplify-and-forward protocol with a fixed gain. Firstly, assuming perfect frequency and phase synchronization, we derive a closed-form error probability approximation for coherent MFSK. Secondly, considering imperfect frequency and phase synchronization, we obtain the exact error probability expressions for noncoherent and coherent BFSK. We have found that a relay network adopting coherent BFSK was more sensitive to the frequency offset than a relay network adopting noncoherent BFSK because the minimum frequency separation of the latter was twice that of the former. Also, we have found that the relay location affected the error performance. With the obtained error probability expressions, it was also possible to analyze the error performance of classical one-hop FSK systems with perfect/imperfect frequency and phase synchronization.

REFERENCES


